

## PROGRESS IN ANISOTROPIC PLASMA PHYSICS

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In 1959 Weibel demonstrated that when a QED plasma has a temperature anisotropy there exist unstable transverse magnetic excitations which grow exponentially fast. In this paper we will review how to determine the growth rates for these unstable modes in the weak-coupling and ultrarelativistic limits in which the collective behavior is describable in terms are so-called “hard-loops”. We will show that in this limit QCD is subject to instabilities which are analogous to the Weibel instability in QED. The presence of such instabilities dominates the early time evolution of a highly anisotropic plasma; however, at longer times it is expected that these instabilities will saturate (condense). We will discuss how the presence of non-linear interactions between the gluons complicates the determination of the saturated state. In order to discuss this we present the generalization of the Braaten-Pisarski isotropic hard-thermal-loop effective action to a system with a temperature anisotropy in the parton distribution functions. The resulting hard-loop effective action can be used to determine the time and energy scales associated with the possible saturation (condensation) of the gluonic modes. We will also discuss the effects of anisotropies on observables, in particular on the heavy quark energy loss.

### 1. Introduction

In the last few years a more or less standard picture of the early stages of a relativistic heavy ion collision has emerged. In its most simplified form there are three assumptions: (1) that the system is boost invariant along the beam direction, (2) that it is homogenous in the directions perpendicular to the beam direction, and (3) that the physics at early times is dominated by

gluons with momentum at a “saturation” scale  $Q_s$  which have occupation numbers of order  $1/\alpha_s$ . The first two assumptions are reasonable for describing the central rapidity region in relativistic heavy-ion collisions. The third assumption relies on the presence of gluonic “saturation” of the nuclear wavefunction at very small values of the Bjorken variable  $x$ <sup>1</sup>. In this regime one can determine the growth of the gluon distribution by requiring that the cross section for deep inelastic scattering at fixed  $Q^2$  does not violate unitarity bounds. As a result the gluon distribution function saturates at a scale  $Q_s$  changing from  $1/k_\perp^2 \rightarrow \log(Q_s^2/k_\perp^2)/\alpha_s$ . Luckily, despite this saturation, due to the factor of  $1/\alpha_s$  in the second scaling relation the occupation number of small- $x$  gluonic modes in the nuclear wavefunction is still large enough to determine their distribution function analytically using classical nonlinear field theory<sup>1</sup>.

These distribution functions are used as input for the subsequent thermalization of the quark-gluon plasma. At short times,  $\tau_0 \sim Q_s^{-1}$ , the gluon distribution function is isotropic, however, after the system begins to expand it rapidly develops a large anisotropy between the transverse and beam directions with  $p_\perp \gg p_z$  due to the fact that initially  $p_z \sim \tau^{-1}$ . In the weak-coupling limit the assumptions above have been used by Baier et al. in an attempt to systematically describe the early stages of quark-gluon plasma evolution in a framework called “bottom-up” thermalization<sup>2</sup>. Using collisional mechanisms they are able to show that during the first stage of evolution hard gluons scatter out-of-plane counteracting the effect of the expansion of the system reducing the rate at which the longitudinal momentum decreases instead to  $p_z \sim Q_s^{2/3} \tau^{-1/3}$ . Although less extreme in terms of the rate at which the longitudinal momenta decreases this scenario still results in considerable momentum-space anisotropies in the gluon distribution function. In such anisotropic systems it has been shown that the physics of the QCD collective modes changes dramatically compared to the isotropic case and instabilities are present which can accelerate the thermalization and isotropization of the plasma<sup>3,4,5,6,8,9,10,11</sup>.

In addition to their role in isotropization and thermalization it is also important to have an understanding of what the impact of such instabilities will be on observables at RHIC and LHC energies. Here we will discuss one test observable – the heavy fermion energy loss<sup>12,13</sup>. One might be worried that there is a fundamental problem with perturbation theory since the presence of instabilities naively causes the calculation of the soft contribution to the heavy quark energy loss to be divergent; however, there is protection mechanism dubbed “dynamical shielding” which renders the

collisional energy loss finite for QED and QCD<sup>12,13</sup>. However, at realistic couplings the presence of instabilities and associated poles on neighboring Riemann sheets<sup>10</sup> causes significant changes in the soft energy loss contribution for both QED and QCD. In fact, as we will discuss, the unphysical poles can even change the sign of the heavy quark energy loss at low momentum turning it instead into energy *gain*.

## 2. Anisotropic Gluon Polarization Tensor

We consider a quark-gluon plasma with a parton distribution function which is decomposed as

$$f(\mathbf{p}) \equiv 2N_f (n(\mathbf{p}) + \bar{n}(\mathbf{p})) + 4N_c n_g(\mathbf{p}) , \quad (1)$$

where  $n$ ,  $\bar{n}$ , and  $n_g$  are the distribution functions of quarks, anti-quarks, and gluons, respectively, and the numerical coefficients collect all appropriate symmetry factors. Using the result of Ref. [6] the spacelike components of the high-temperature gluon self-energy for gluons with soft momentum ( $k \sim gT$ ) can be written as

$$\Pi^{ij}(K) = -\frac{g^2}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} v^i \partial^l f(\mathbf{p}) \left( \delta^{jl} + \frac{v^j k^l}{K \cdot V + i\epsilon} \right) , \quad (2)$$

where the parton distribution function  $f(\mathbf{p})$  is, in principle, completely arbitrary. In what follows we will assume that  $f(\mathbf{p})$  can be obtained from an isotropic distribution function by the rescaling of only one direction in momentum space. In practice this means that, given any isotropic parton distribution function  $f_{\text{iso}}(p)$ , we can construct an anisotropic version by changing the argument of the isotropic distribution function

$$f(\mathbf{p}) = \sqrt{1 + \xi} f_{\text{iso}} \left( \sqrt{\mathbf{p}^2 + \xi(\mathbf{p} \cdot \hat{\mathbf{n}})^2} \right) , \quad (3)$$

where the factor of  $\sqrt{1 + \xi}$  is a normalization constant which ensures that the same parton density is achieved regardless of the anisotropy introduced,  $\hat{\mathbf{n}}$  is the direction of the anisotropy, and  $\xi > -1$  is an adjustable anisotropy parameter with  $\xi = 0$  corresponding to the isotropic case. Here we will concentrate on  $\xi > 0$  which corresponds to a contraction of the distribution along the  $\hat{\mathbf{n}}$  direction since this is the configuration relevant for heavy-ion collisions at early times, namely two hot transverse directions and one cold longitudinal direction.

Making a change of variables in (2) it is possible to integrate out the  $|p|$ -dependence giving <sup>6</sup>

$$\Pi^{ij}(\omega/k, \theta_n) = \mu^2 \int \frac{d\Omega}{4\pi} v^i \frac{v^l + \xi(\mathbf{v} \cdot \hat{\mathbf{n}})\hat{n}^l}{(1 + \xi(\mathbf{v} \cdot \hat{\mathbf{n}})^2)^2} \left( \delta^{jl} + \frac{v^j k^l}{K \cdot V + i\epsilon} \right), \quad (4)$$

where  $\cos \theta_n \equiv \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}$  and  $\mu^2 \equiv \sqrt{1 + \xi} m_D^2 > 0$ . The isotropic Debye mass,  $m_D$ , depends on  $f_{\text{iso}}$ . In the case of pure-gauge QCD with an equilibrium  $f_{\text{iso}}$  we have  $m_D = gT$ .

The next task is to construct a tensor basis for the spacelike components of the gluon self-energy and propagator. We therefore need a basis for symmetric 3-tensors which depend on a fixed anisotropy 3-vector  $\hat{n}^i$  with  $\hat{n}^2 = 1$ . This can be achieved with the following four component tensor basis:  $A^{ij} = \delta^{ij} - k^i k^j / k^2$ ,  $B^{ij} = k^i k^j / k^2$ ,  $C^{ij} = \tilde{n}^i \tilde{n}^j / \tilde{n}^2$ , and  $D^{ij} = k^i \tilde{n}^j + k^j \tilde{n}^i$  with  $\tilde{n}^i \equiv A^{ij} \hat{n}^j$ . Using this basis we can decompose the self-energy into four structure functions  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  as  $\mathbf{\Pi} = \alpha \mathbf{A} + \beta \mathbf{B} + \gamma \mathbf{C} + \delta \mathbf{D}$ .<sup>a</sup>

### 3. Collective Modes

As shown in Ref. [6] this tensor basis allows us to express the propagator in terms of the following three functions

$$\begin{aligned} \Delta_{\alpha}^{-1}(K) &= k^2 - \omega^2 + \alpha, \\ \Delta_{\pm}^{-1}(K) &= \omega^2 - \Omega_{\pm}^2, \end{aligned}$$

where  $2\Omega_{\pm}^2 = \bar{\Omega}^2 \pm \sqrt{\bar{\Omega}^4 - 4((\alpha + \gamma + k^2)\beta - k^2 \tilde{n}^2 \delta^2)}$  and  $\bar{\Omega}^2 = \alpha + \beta + \gamma + k^2$ .

Taking the static limit of these three propagators we find that there are three mass scales:  $m_{\pm}$  and  $m_{\alpha}$ . In the isotropic limit,  $\xi \rightarrow 0$ ,  $m_{\alpha}^2 = m_{-}^2 = 0$  and  $m_{+}^2 = m_D^2$ . However, for  $\xi > 0$  we find that  $m_{\alpha}^2 < 0$  for all  $|\theta_n| \neq \pi/2$  and  $m_{-}^2 < 0$  for all  $|\theta_n| \leq \pi/4$ . Note also that for  $\xi > 0$  both  $m_{\alpha}^2$  and  $m_{-}^2$  have their largest negative values at  $\theta_n = 0$  where they are equal.

The fact that for  $\xi > 0$  both  $m_{\alpha}^2$  and  $m_{-}^2$  can be negative indicates that the system is unstable to both magnetic and electric fluctuations with the fastest growing modes focused along the beamline ( $\theta_n = 0$ ). In fact it can be shown that there are two purely imaginary solutions to each of the dispersion relations  $\Delta_{\alpha}^{-1}(K) = 0$  and  $\Delta_{-}^{-1}(K) = 0$  with the solutions in the upper half plane corresponding to unstable modes. We can determine

<sup>a</sup>Explicit analytic integral expressions for  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  can be found in Ref. [6]. Additionally, analytic expressions in the small- $\xi$  limit and for propagation along the anisotropy direction can be found in Refs. [6] and [10], respectively.

the growth rate for these unstable modes by taking  $w \rightarrow i\Gamma$  and then solving the resulting dispersion relations for  $\Gamma(k)$ .

In Fig. 1 we plot the instability growth rates,  $\Gamma_\alpha$  and  $\Gamma_-$ , as a function of wave number for  $\xi = 10$  and  $\theta_n = \pi/8$ . Note that both growth rates vanish at  $k = 0$  and have a maximum  $\Gamma_* \sim \mu/10$  at  $k_* \sim \mu/3$ . The fact that they have a maximum means that at early times the system will be dominated by unstable modes with spatial frequency  $1/k_*$ .

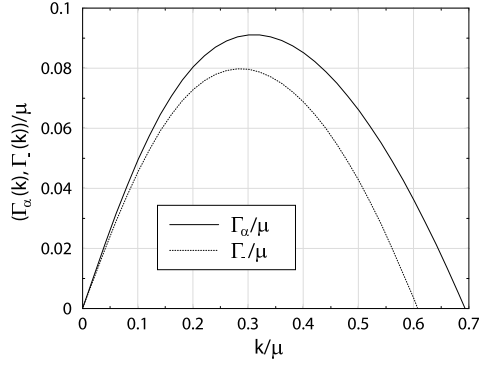


Fig. 1: Instability growth rates.

#### 4. The Hard-Loop Effective Action

In the previous section by only considering the polarization tensor we implicitly used a linear analysis which suffices only during very early times. As the instabilities grow non-linear interactions become more important and can halt their growth. In QED simulations such a saturated state does emerge and is called a Bernstein-Greene-Kruskal wave; however, in QCD any corresponding saturated state would necessarily be a much more complex beast. Answering these questions requires knowledge of the full hard-loop effective Lagrangian in an anisotropic system. Luckily, it is possible to obtain a simple expression which generates all hard-loop (HL) vertex functions<sup>9</sup>

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g^2}{2} \int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|} F_{\mu\nu}(x) \frac{p^\rho p^\nu}{(p \cdot D)^2} F_\rho{}^\mu(x) \quad (5)$$

For example, from this we can obtain the hard-loop 3-gluon vertex

$$\Gamma_{\text{HL}}^{\mu\nu\lambda}(k, q, r) = \frac{g^2}{2} \int_{\mathbf{p}} \frac{\partial f(\mathbf{p})}{\partial p^\beta} \hat{p}^\mu \hat{p}^\nu \hat{p}^\lambda \left( \frac{r^\beta}{\hat{p} \cdot q \hat{p} \cdot r} - \frac{k^\beta}{\hat{p} \cdot k \hat{p} \cdot q} \right). \quad (6)$$

In contrast to the isotropic  $n$ -gluon HL vertices which vanish in the static limit, the anisotropic  $n$ -gluon HL vertices can be non-zero. This opens up the possibility for additional saturation mechanisms which are governed by soft non-abelian physics sensitive to these higher vertices.

Note that Arnold and Lenaghan have shown that if one considers fluctuations in the vector potential which are directed along the anisotropy

direction and only depend on the longitudinal coordinate then it is possible to reduce the effective lagrangian (5) to a quadratic form which simplifies the analysis considerably<sup>11</sup>. They then make some additional assumptions including ignoring time-dependence in the self-energy in order to construct a toy-model which is then used to study the subsequent evolution of instabilities in the system. Their results indicate that in non-abelian gauge theories that possible non-abelian saturated states are metastable and as a result instability growth forces the system along abelian directions. However, it should be noted that by ignoring the time dependence in the potential as they have done, the instability growth rate in their model is finite at  $k = 0$ . This is not the case when time dependence is included in the full analysis and instead one finds that the growth rate at  $k = 0$  then vanishes as shown in Fig. 1. This difference could have a significant impact on the abelianization particularly on the “abelianization length” measured in their simulations.

## 5. Effects on observables

Since the unstable modes manifest themselves as poles of the propagator in the static limit, it has been argued<sup>14</sup> that the presence of these instabilities in general prohibits the calculation of observables in a perturbative framework, since those quantities would be plagued by unregulated divergences.

However, it turns out that at least one observable, namely the collisional energy loss of a heavy fermion, is protected from these unregulated divergencies by a mechanism dubbed “dynamical shielding”<sup>12</sup> (see Ref. [7] for a proof in the case of  $\xi \ll 1$  and  $\xi \rightarrow \infty$ ). As a consequence of dynamical shielding (which incidently is somewhat similar to dynamical screening of the magnetic interaction in QCD, hence the name), the collisional energy loss becomes calculable perturbatively also in the anisotropic case. Therefore, we will use the collisional energy loss as “test observable” to learn something about the effects of the instabilities on physical quantities.

Schematically, the soft contribution to the collisional energy loss is given by

$$-\left(\frac{dW}{dt}\right)_{\text{soft}} \sim \text{Im} \int \frac{d^3\mathbf{k}}{(2\pi)^3} k z \Delta(z, k, \theta_n), \quad (7)$$

where  $\Delta(z, k, \theta_n) = v^i \Delta^{ij} v^j$  is the relevant contraction of the propagator<sup>12</sup> and  $z = \hat{\mathbf{k}} \cdot \mathbf{v}$  with  $\mathbf{v}$  the velocity of the heavy fermion. The instabilities would in principle affect the integrand for  $\text{Re } z = 0$  causing singularities to be encountered along the momentum integration path. In order to see that

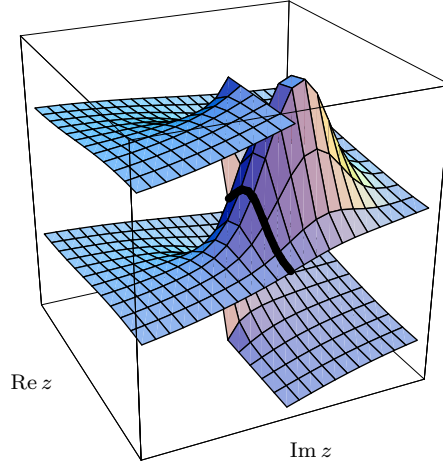


Fig. 2: Sketch of the complex  $z$  plane including the extension of the logarithm to the unphysical sheet. Also shown is how a pole in the unphysical region (mountain) has effects felt on the physical sheet. The black line indicates where the two sheets are joined together.

there are, in fact, no singularities we focus on terms of the form  $(q^2 + \alpha)^{-1}$ . Due to the fact that in the static limit the structure function  $\alpha$  is negative-valued we could encounter a singularity at  $q^2 = \alpha$ ; however, if one takes the static limit of  $\alpha$  carefully we find

$$\lim_{z \rightarrow 0} \alpha(z) = M^2(-1 + iDz) + \mathcal{O}(z^2), \quad (8)$$

where  $M$  and  $D$  depend on the angle of propagation with respect to the anisotropy vector and the strength of the anisotropy. As long as  $D$  is non-vanishing the singularities are regularized because of the  $z$  in the numerator of Eq. (7) and we call the singularity “dynamically shielded”. This effect can be shown to regulate all singularities which would have naively come from the presence of instabilities. Therefore, there is no catastrophic effect from the instabilities on the collisional energy loss. However, there is an effect which is associated with the presence of the instabilities; in order to understand this effect it is however necessary to extend the analysis of the collective modes of an anisotropic plasma to arbitrary Riemann sheets.

The extension to the unphysical Riemann sheets is achieved by continuing analytically the structure functions beyond their cut structures (which for finite  $\xi$  is a logarithmic cut running along the real  $z$  axis for  $z^2 < 1$ , in

complete analogy to the isotropic case, but a square-root cut for  $z^2 < \sin^2 \theta_n$  for  $\xi \rightarrow \infty$ )<sup>10</sup>. More precisely, since one can continue the structure functions from either below or above the cut, the physical sheet has two neighboring unphysical Riemann sheets, which – at least in the case of finite  $\xi$  – could themselves be extended to reproduce the well-known “spiral staircase” form of the complex logarithm. Once the structure functions on the unphysical sheets are known, one may conduct the analysis of the collective modes as was done on the physical sheet, finding – among others – singularities for complex  $z$  on the neighboring unphysical sheets that extend to the region of spacelike momenta<sup>10</sup>.

But why bother about these modes, given that they do not “live” on the physical sheet? To see that there can be an effect on physical quantities, imagine once again the spiral staircase spanned by the complex logarithm: the physical sheet would correspond to the region covered by the spiral plane from the ground floor to the first floor, while the unphysical sheet where the extra quasiparticle mode lives would correspond to the region first to second floor. Since the existence of such a mode corresponds to a pole in the propagator, there is has a mountainous dent (singularity) in the spiral plane somewhere from the first to the second floor. However, since the mountain has a finite width, its base can be felt also below the second floor, especially if its peak is near the first floor (see Fig. 2 for a sketch). Therefore, the nearer a mode on a neighboring unphysical sheet comes to the border of the physical sheet, the more will it affect the propagator on the physical sheet and consequently any physical observable sensitive to the relevant region of phase-space (in this case spacelike momentum).

This is precisely the situation encountered for the collisional energy loss: there are unstable modes on the physical sheet which, once the momentum  $k$  becomes larger than some critical value (see Fig. 1), move onto the neighboring unphysical sheets and subsequently influence the propagator on the physical sheet by the mechanism explained above. In fact, it turns out that the contributions from these unphysical modes may drive the leading-order perturbative results for the collisional energy loss *negative* for small fermion velocities, corresponding to an energy transfer from hard to soft scales. Note, however, that this negative energy loss is *not* due to the heavy fermion in question having a sub-thermal velocity but is instead connected to the instabilities of an anisotropic system, as already noted in Ref. [15].

For further discussion of the heavy quark energy loss in an anisotropic quark-gluon plasma we refer the reader to Ref. [13]. In addition to elucidating the impact of unstable modes and associated poles on unphysical



Riemann sheets, in that paper we showed that for anisotropic systems the heavy quark collisional energy loss has an angular dependence which increases as the coupling and/or anisotropy is increased. Quantitatively, for  $\alpha_s = 0.3$  and a 20 GeV bottom quark we found that the deviations from the isotropic result were on the order of 10% for  $\xi = 1$  and of the order of 20% for  $\xi \geq 10$ . When translated into the difference between longitudinal and transverse energy loss this results in a 10% difference at  $\xi = 1$ , a 30% difference at  $\xi = 10$ , and a 50% difference at  $\xi = \infty$ .

## 6. Conclusions

In this paper we have attempted to summarize the developments which have occurred in the last two years regarding the physics of a quark-gluon plasma which has a momentum-space anisotropy in the underlying parton distribution functions. We have presented expressions for the self-energy and full effective action in this case and discussed the emergence of unstable modes which cause exponential growth of the gluon field along the anisotropy direction. In addition, we discussed the expected effects such anisotropies have on observables concentrating on the heavy quark energy loss as a test observable.

As we have tried to illustrate here the physics of anisotropic plasmas is qualitatively different from isotropic ones. In the weak-coupling limit the short-time behaviour of anisotropic plasmas is dominated by the growth of unstable modes. In order to properly understand thermalization and isotropization of a quark-gluon plasma it seems necessary to take into account the effect of such unstable modes from the earliest times. We note in closing if a Vlasov description is applicable at very early times ( $\tau_0 \sim Q_s^{-1}$ ) then instability broadening of the longitudinal momentum is much more effective than collisional broadening and it is possible that anisotropies due to expansion of the system are, in fact, never generated or reduced significantly. However, a more conservative approach calls for the Vlasov description to only be applied once the hard gluon occupation number becomes less than one which according to collisional arguments occurs at a time  $\tau_1 \sim \alpha^{-3/2} Q_s^{-1}$ . If this more conservative approach is applicable then the anisotropy at  $\tau_1$  can be parametrically estimated to be  $\xi \sim \alpha_s^{-3/2}$  ( $\xi \sim \alpha_s^{-1/2}$  if collisional broadening is taken in account) which results in a strongly anisotropic system in the weak-coupling limit. Whichever approach is used, however, it is clear that it is necessary to take into account the unstable modes.

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## References

1. L. V. Gribov, E. M. Levin and M. G. Ryskin, Phys. Rept. **100**, 1 (1983); A. H. Mueller and J. w. Qiu, Nucl. Phys. B **268**, 427 (1986); J. P. Blaizot and A. H. Mueller, Nucl. Phys. B **289**, 847 (1987); L. McLerran and R. Venugopalan, Phys. Rev. D **49**, 223 (1994); **49**, 3352 (1994); **50**, 2225 (1994); I. Balitskii, Nucl. Phys. B **463**, 99 (1996); J. Jalilian-Marian, A. Kovner, L. McLerran and H. Weigert, Phys. Rev. D **55**, 5414 (1997); Yu.V. Kovchegov and A.H. Mueller, Nucl. Phys. B **529**, 451 (1998); Yu.V. Kovchegov, Phys. Rev. D **60**, 034008 (1999); **61**, 074018 (2000).
2. R. Baier, A.H. Mueller, D. Schiff, and D.T. Son, Phys. Lett. B **502**, 51 (2001).
3. E.S. Weibel, Phys. Rev. Lett. **2**, 83 (1959).
4. S. Mrówczyński, Phys. Lett. B **314**, 118 (1993); Phys. Rev. C **49**, 2191 (1994); Phys. Lett. B **393**, 26 (1997).
5. J. Randrup and S. Mrówczyński, Phys. Rev. C **68**, 034909 (2003).
6. P. Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003).
7. P. Romatschke, “Quasiparticle description of the hot and dense quark gluon plasma,” PhD Dissertation, arXiv: hep-ph/0312152, (2003).
8. P. Arnold, J. Lenaghan, and G. D. Moore, JHEP **0308**, 002 (2003).
9. S. Mrówczyński, A. Rebhan, and M. Strickland, Phys. Rev. D **70**, 025004 (2004).
10. P. Romatschke and M. Strickland, arXiv: hep-ph/0406188, (2004).
11. P. Arnold and J. Lenaghan, arXiv: hep-ph/0408052, (2004).
12. P. Romatschke and M. Strickland, Phys. Rev. D **69**, 065005 (2004).
13. P. Romatschke and M. Strickland, arXiv: hep-ph/0408275, (2004).
14. G. Moore. QCD Kinetic theory (lecture), Quantum fields in and out of equilibrium, University of Bielefeld, (2003).
15. E.M Lifshitz and L.P. Pitaevskii, “Physical kinetics”, Pergamon Press, Oxford, §30 (1981).